

Some new results on Lagrange interpolation for bounded variation functions

G. Mastroianni*, M.G. Russo

*Department of Mathematics and Computer Science, University of Basilicata, v.le dell'Ateneo Lucano 10,
85100 Potenza, Italy*

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Abstract

The paper deals with the Lagrange interpolation of functions having a bounded variation derivative. For special systems of nodes, it is shown that this polynomial sequence converges with the best approximation order. The L^p weighted case is also discussed.

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1. Introduction

In a previous paper [4] it was proved that, for every continuous function f in $(-1, 1)$, suitable matrices of nodes exist such that the corresponding sequence of Lagrange interpolating polynomials converges to the function f with order $o\left(m^{-\frac{1}{p}}\right)$, in some L^p weighted space, $1 < p < \infty$.

In this paper we prove that a similar result holds true if the function f is of bounded variation ($f \in \mathcal{BV}$) and without requiring its continuity. Moreover the order of convergence is the best. This is shown in [Theorem 2.1](#). A consequence of this theorem is [Corollary 2.1](#), where continuous functions of bounded variation are considered.

* Corresponding author.

E-mail addresses: giuseppe.mastroianni@unibas.it (G. Mastroianni), mariagrazia.russo@unibas.it (M.G. Russo).

Moreover we will study the behavior, in uniform norm, of the Lagrange polynomials of continuous functions having derivatives of bounded variation (eventually discontinuous). Denoting by $\omega(\cdot, t)_p$ the ordinary modulus of continuity, if $f' \in \mathcal{BV}$ and is discontinuous then it results that $\omega(f', t)_\infty \geq C > 0$ and $\omega(f', t)_p \sim t^{\frac{1}{p}}$. In this sense the \mathcal{BV} condition, differently from the L^p case, is not a smoothness condition for f' w.r.t. the sup-norm. Nevertheless this condition produces positive effects on the convergence of the interpolatory processes.

There is a wide literature on the topic and several authors proved Jackson type estimates for continuous functions of bounded variation. For the case of the trigonometric interpolation, we recall the paper by Nevai [9]. For the algebraic case P. Vértési showed, in [15,16], that if f is continuous and of bounded variation then, for some matrices of knots, the sequence of the interpolating polynomials uniformly converges to f on the interval $[-1, 1]$. This result was extended to different classes of functions and we recall, among others, [12,13,5,6,3] and the references therein.

In this paper we consider the interpolating processes based on matrices of nodes satisfying a special condition (see (2.21) in Section 2.2). Using this type of interpolation, we prove a Jackson type theorem for the case of functions with the r th derivative of bounded variation. The proofs are very simple and the order of convergence is the best possible. As an example, we will show that a wide class of matrices of nodes satisfy the condition (2.21).

2. Main results

Let us introduce some notation. In the sequel frequently we shall denote by C a positive constant which may be different in different formulas. If C is a constant independent of (dependent on) the parameters a, b, \dots , sometimes we shall write $C \neq C(a, b, \dots)$ ($C = C(a, b, \dots)$). If A and B are two positive quantities, depending on some parameters, then $A \sim B$ means that a positive constant M , independent of the parameters of A and B , exists such that $(\frac{A}{B})^{\pm 1} \leq M$. Moreover we will denote by \mathbb{P}_m the set of all polynomials of degree at most m .

Let $w(x) = v^{\alpha, \beta}(x) = (1-x)^\alpha(1+x)^\beta$, $\alpha, \beta > -1$ and $\{p_m(w)\}_{m=0,1,\dots}$ be the sequence of the orthonormal Jacobi polynomials having positive leading coefficients. Denote by $L_m(w, f) \in \mathbb{P}_{m-1}$ the Lagrange polynomial interpolating a function f on the zeros $-1 < x_1 < \dots < x_m < 1$, $x_k \equiv x_{m,k}(w)$, of $p_m(w)$, i.e.

$$L_m(w, f, x) = \sum_{k=1}^m l_k(x) f(x_k), \quad l_{m,k}(x) = \frac{p_m(w, x)}{p'_m(w, x_k)(x - x_k)}. \quad (2.1)$$

When f is of bounded variation ($f \in \mathcal{BV}$) and x_k , for some k , is a jump point for f , we set

$$f(x_k) := \frac{f(x_k)^- + f(x_k)^+}{2}, \quad (2.2)$$

where $f(x_k)^\pm = \lim_{x \rightarrow x_k^\pm} f(x)$.

We also recall that a function f , with $f^{(r)} \in \mathcal{BV}$, admits the following representation:

$$f(x) = T_r(f, x) + \int_{-1}^1 \Gamma_{l,r}(x) df^{(r)}(t) \quad (2.3)$$

where $T_r(f) \in \mathbb{P}_r$ is the Taylor polynomial of f , with -1 as the starting point, $\Gamma_{l,r}(x) := \frac{(x-t)_+^r}{r!}$ and

$$(x-t)_+^r = \begin{cases} (x-t)^r, & x > t \\ 0, & x \leq t \end{cases}$$

denotes the r th truncated power.

2.1. The L^p case

We define the space L^p , $1 \leq p < \infty$, in the usual way and, if $u(x) = v^{\gamma, \delta}(x)$, $\gamma, \delta > -1$, is a Jacobi weight, we will write $f \in L_u^p$ iff $fu \in L^p$.

In a former paper [4] the authors proved the following theorem.

Theorem A. Let $f \in L_u^p$, $1 < p < \infty$. If f is continuous on $(-1, 1)$ ($f \in C^0(-1, 1)$) the following inequality:

$$\| [f - L_m(w, f)]u \|_p \leq \frac{C}{m^{\frac{1}{p}}} \int_0^{\frac{1}{m}} \frac{\Omega_\varphi^s(f, t)_{u,p}}{t^{1+\frac{1}{p}}} dt, \quad C \neq C(f, m), \quad (2.4)$$

holds true if

$$\frac{u}{\sqrt{w\varphi}} \in L^p, \quad \frac{\sqrt{w\varphi}}{u} \in L^q, \quad q = \frac{p}{p-1} \quad (2.5)$$

where $\Omega_\varphi^s(f, t)_{u,p}$, $\varphi(t) = \sqrt{1-t^2}$, is the main part of the s th modulus of smoothness of Ditzian and Totik. Moreover if the integral $\int_0^1 \frac{\Omega_\varphi^s(f, t)_{u,p}}{t^{1+\frac{1}{p}}} dt$ is bounded then (2.5) are also necessary conditions.

By (2.4), with (2.5) holding, if $\int_0^1 \frac{\Omega_\varphi^s(f, t)_{u,p}}{t^{1+\frac{1}{p}}} dt < +\infty$, then it follows that

$$\| [f - L_m(w, f)]u \|_p = o\left(m^{-\frac{1}{p}}\right).$$

Now if we replace the assumption $f \in C^0(-1, 1)$ with $f \in \mathcal{BV}$, we obtain the following theorem.

Theorem 2.1. Let $f \in L_u^p$, $1 < p < \infty$, be such that $f^{(r)} \in \mathcal{BV}$, $r \geq 0$. Then there exists a constant C , independent of m and f , such that the estimate

$$\| [f - L_m(w, f)]u \|_p \leq \frac{C}{m^{r+\frac{1}{p}}} \int_{-1}^1 \left(\sqrt{1-t^2}\right)^{r+\frac{1}{p}} u(t) |df^{(r)}(t)|, \quad m > r, \quad (2.6)$$

holds if and only if the weights u, w satisfy (2.5).

Note that for $r = 0$, i.e. for $f \in \mathcal{BV}$, it results that $\Omega_\varphi(f, t)_{u,p} \sim t^{\frac{1}{p}}$ and hence we cannot use (2.4). We also remark that under the assumptions of Theorem 2.1, $L_m(w, f)$ behaves like the polynomial of best approximation (see for instance [14, p. 412]). The theorem refines and extends some results from [11].

Proof. We first prove that (2.6) implies (2.5).

Let $r = 0$. Since (2.6) holds true for any function $f \in \mathcal{BV}$, we choose a bounded variation function g s.t. $|g(x)| \leq 1$ and $|dg(x)| \leq 1$ in order to get, by (2.6),

$$\| L_m(w, g)u \|_p \leq C \neq C(m).$$

On the other hand, by a result in [7], it follows that

$$C_1 \left\| \frac{u}{\sqrt{w\varphi}} \right\|_p \leq C \|p_m(w)u\|_p \leq \|L_m(w, g)u\|_p \leq C_2$$

where C, C_1 and C_2 are absolute positive constants, and hence the first condition in (2.5) is satisfied.

Now we prove that condition $\frac{\sqrt{w\varphi}}{u} \in L^q$ is also necessary. Let m be fixed and $d \in \{1, \dots, m\}$. Set

$$f_d(x) := \begin{cases} \frac{x - x_{d-1}}{x_d - x_{d-1}}, & \text{if } x \in [x_{d-1}, x_d] \\ \frac{x_{d+1} - x}{x_{d+1} - x_d}, & \text{if } x \in [x_d, x_{d+1}] \\ 0, & \text{elsewhere.} \end{cases}$$

By (2.6), with $r = 0$, it follows that

$$\begin{aligned} \|L_m(w, f_d)u\|_p &\leq \|f_d u\|_p + C \int_{-1}^1 \left(\frac{\varphi(t)}{m} \right)^{\frac{1}{p}} u(t) |df_d(t)| \\ &\leq C \left(\int_{x_{d-1}}^{x_{d+1}} u^p(x) dx \right)^{\frac{1}{p}} + C \left(\frac{\varphi(x_d)}{m} \right)^{\frac{1}{p}} u(x_d) \\ &\sim \left(\frac{\varphi(x_d)}{m} \right)^{\frac{1}{p}} u(x_d) \sim \lambda_m^{\frac{1}{p}}(u^p, x_d) \end{aligned} \quad (2.7)$$

where $\lambda_m^{\frac{1}{p}}(u^p, x)$ denotes the m th Christoffel function related to the weight u^p . Therefore, since $L_m(w, f_d, x) \equiv l_{m,d}(w, x)$, where $l_{m,d}(w)$ is defined in (2.1), we get

$$\|l_{m,d}(w)u\|_p = \|L_m(w, f_d)u\|_p \leq C \lambda_m^{\frac{1}{p}}(u^p, x_d).$$

In conclusion, following step by step the proof in [4, pp. 278–279], we deduce that $\frac{\sqrt{w\varphi}}{u} \in L^q$.

Now we prove that conditions (2.5) are sufficient for obtaining (2.6). Let us first assume that (2.6) is true for $r = 0$, i.e. we suppose $f \in \mathcal{BV}$ and, for any couple u, w of weights, that conditions (2.5) imply the estimate

$$\|[f - L_m(w, f)]u\|_p \leq \frac{C}{m^{\frac{1}{p}}} \int_{-1}^1 \varphi^{\frac{1}{p}}(t) u(t) |df(t)|, \quad (2.8)$$

with $C \neq C(m, f)$ and $1 < p < \infty$.

Let $f' \in \mathcal{BV}$ in $(-1, 1)$ and consequently $\|f'\varphi u\|_p < \infty$. By (2.4) we get

$$R_m(f)_u := \|[f - L_m(w, f)]u\|_p \leq \frac{C}{m} \left(\int_{-1}^1 |f'(t)\varphi(t)u(t)|^p dt \right)^{\frac{1}{p}},$$

and hence, with $Q_{m-1}(x) = \int_{-1}^x L_{m-2}(w\varphi^2, f', y)dy$, it follows that

$$\begin{aligned} R_m(f)_u &= R_m(f - Q_{m-1})_u \\ &\leq \frac{C}{m} \left(\int_{-1}^1 \left| [f'(t) - L_{m-2}(w\varphi^2, f', x)] \varphi(t)u(t) \right|^p dt \right)^{\frac{1}{p}}. \end{aligned}$$

Using (2.8), we can estimate the right-hand side by replacing f by f' , w by $w\varphi^2$ and u by $u\varphi$. Since $w\varphi^2$ and $u\varphi$ surely satisfy (2.5), we conclude that

$$\| [f - L_m(w, f)]u \|_p \leq \frac{C}{m^{1+\frac{1}{p}}} \int_{-1}^1 \varphi^{1+\frac{1}{p}}(t) u(t) |df'(t)|, \quad (2.9)$$

if $f' \in \mathcal{BV}$. Therefore the theorem follows by induction on r , if we prove (2.8). To this end, if $f \in \mathcal{BV}$, then from (2.3) and the generalized Minkowski inequality it follows that

$$\| [f - L_m(w, f)]u \|_p \leq \int_{-1}^1 \| [\Gamma_{t,0} - L_m(w, \Gamma_{t,0})]u \|_p |df(t)|. \quad (2.10)$$

We remark that, since $\Omega_\varphi(\Gamma_{t,0}, \tau)_{u,p} \sim \tau^{\frac{1}{p}} \varphi^{\frac{1}{p}}(t) u(t)$, we cannot use (2.4) in order to estimate the norm at the right-hand side of (2.10).

The theorem follows by (2.10) and the following proposition. \square

Proposition 2.1. *Let u, w be two Jacobi weights with $u \in L^p$. If conditions (2.5) are satisfied then, for $1 < p < \infty$ and $t \in [-1, 1]$, a constant C exists, independent of m and t , such that*

$$\| [\Gamma_{t,0} - L_m(w, \Gamma_{t,0})]u \|_p \leq C \left(\frac{\varphi(t)}{m} \right)^{\frac{1}{p}} u(t) \quad (2.11)$$

holds true.

Proof. We note that if $t \in [-1, x_1] \cup [x_m, 1]$ then

$$\| [\Gamma_{t,0} - L_m(w, \Gamma_{t,0})]u \|_p = \left(\int_{-1}^t u^p(x) dx + \int_t^1 u^p(x) dx \right)^{\frac{1}{p}} \leq C u(t) \left(\frac{\varphi(t)}{m} \right)^{\frac{1}{p}}$$

where C is a positive constant independent of t and m .

If $t \in [x_1, x_m]$, there exists $d \in \{1, \dots, m\}$ s.t. $x_d \leq t \leq x_{d+1}$, and we define

$$F_{t,m}(x) := \begin{cases} 0, & \text{if } x < x_{d-1} \\ \frac{x-t}{t-x_{d-1}} + 1, & \text{if } x \in [x_{d-1}, t] \\ 1, & \text{if } x > t \end{cases} \quad (2.12)$$

where, if $d = 1$, $x_{d-1} := -1$. Hence we get

$$\begin{aligned} \| [\Gamma_{t,0} - L_m(w, \Gamma_{t,0})]u \|_p &\leq \| [\Gamma_{t,0} - F_{t,m}]u \|_p + \| [F_{t,m} - L_m(w, F_{t,m})]u \|_p \\ &\quad + \| L_m(w, \Gamma_{t,0} - F_{t,m})u \|_p =: A_1 + A_2 + A_3. \end{aligned} \quad (2.13)$$

Evaluate A_i , $i = 1, 2, 3$, separately.

By the definition of Γ_t and $F_{t,m}$, for any fixed $t \in [x_1, x_m]$, it follows that

$$\begin{aligned} A_1 &:= \| [\Gamma_{t,0} - F_{t,m}]u \|_p \leq \left(\int_{x_{d-1}}^t F_{t,m}^p(x) u^p(x) dx \right)^{\frac{1}{p}} \\ &\leq \left(\int_{x_{d-1}}^t u^p(x) dx \right)^{\frac{1}{p}} \sim u(t) \left(\frac{\varphi(t)}{m} \right)^{\frac{1}{p}}. \end{aligned} \quad (2.14)$$

In order to evaluate A_2 we remark that $F_{t,m}$ is absolutely continuous on $(-1, 1)$ and that $\| F'_{t,m} \varphi u \|_p < C$. Therefore it is possible to apply Theorem A, i.e. under the assumptions (2.5)

we get

$$A_2 := \|[F_{t,m} - L_m(w, F_{t,m})]u\|_p \leq \frac{C}{m} \|F'_{t,m} \varphi u\|_p$$

where C is independent of m and t . Therefore we have

$$\begin{aligned} A_2 &\leq \frac{C}{m} \left(\int_{x_{d-1}}^t |F'_{t,m}(x)|^p \varphi^p(x) u^p(x) dx \right)^{\frac{1}{p}} \\ &= \frac{C}{m} \left(\frac{u^p(t) \varphi^p(t)}{(t - x_{d-1})^{p-1}} \right)^{\frac{1}{p}} \sim u(t) \left(\frac{\varphi(t)}{m} \right)^{\frac{1}{p}}. \end{aligned} \quad (2.15)$$

It remains to deduce an estimate for A_3 . We remark that if we denote by $l_{m,k}(w, x)$, $k = 1, \dots, m$, the fundamental Lagrange polynomials, by the definition of $\Gamma_{t,0}$ and $F_{t,m}$ we get

$$\begin{aligned} L_m(w, \Gamma_{t,0} - F_{t,m}, x) &= \sum_{k=1}^m l_{m,k}(x) (\Gamma_{t,0} - F_{t,m})(x_k) \\ &= -l_{m,d}(x) F_{t,m}(x_d) = l_{m,d}(x) \left(\frac{t - x_d}{t - x_{d-1}} - 1 \right). \end{aligned} \quad (2.16)$$

Therefore, if $1 < p < \infty$, by applying the Marcinkiewicz inequality [4] (that holds true under the assumptions (2.5)), we get

$$A_3 := \|L_m(w, \Gamma_{t,0} - F_{t,m})u\|_p \leq \|l_{m,d}(w)u\|_p \leq C \lambda_m^{\frac{1}{p}}(u^p, x_d),$$

$\lambda_m(u^p, x)$ denoting the m th Christoffel function related to the weight u^p . Since it is known that $\lambda_m(u^p, x_d) \sim \frac{\sqrt{1-x_d^2}}{m} u^p(x_d)$ (see for instance [10]) we have

$$A_3 \leq C \lambda_m^{\frac{1}{p}}(u^p, x_d) \sim C u(t) \left(\frac{\varphi(t)}{m} \right)^{\frac{1}{p}}, \quad (2.17)$$

where $x_d \leq t \leq x_{d+1}$ and where C is independent of m and t . \square

Theorem 2.1 ensures that, under the assumptions $f^{(r)} \in \mathcal{BV}$ and (2.5), it results that

$$\|[f - L_m(w, f)]u\|_p = \mathcal{O}\left(m^{-r-\frac{1}{p}}\right).$$

For more than one context it is interesting to state under which assumptions on f the “ \mathcal{O} ” can be replaced by “ o ”. A simple L^p condition follows by **Theorem A**. Indeed if

$$\int_0^1 \frac{\Omega_\varphi^s(f^{(r)}, t)_{u,p}}{t^{1+\frac{1}{p}}} dt < +\infty, \quad s > r, \quad (2.18)$$

then by (2.4) it easily follows that

$$\|[f - L_m(w, f)]u\|_p = o\left(m^{-r-\frac{1}{p}}\right).$$

Note that (2.18) implies that $f^{(r)} \in C^0(-1, 1)$ (see [2]). But we get the following corollary.

Corollary 2.1. Under the assumptions (2.5), if $f^{(r)} \in \mathcal{BV}$, $r \geq 0$, is continuous on $[-1, 1]$, then

$$\| [f - L_m(w, f)]u \|_p = o\left(m^{-r-\frac{1}{p}}\right).$$

and the constants in “ o ” are independent of m .

Proof. Let $r = 0$. Estimate (2.6) with $r = 0$ gives

$$\|(R_m f)u\|_p := \|[f - L_m(w, f)]u\|_p \leq \frac{C}{m^{\frac{1}{p}}} \int_{-1}^1 (\varphi^{\frac{1}{p}} u)(t) |df(t)|. \quad (2.19)$$

Since $f \in \mathcal{BV}$ and is a continuous function, then $f' \in L^1$ [8, pp. 246, 250]. Therefore

$$\|(R_m f)u\|_p \leq \frac{C}{m^{\frac{1}{p}}} \int_{-1}^1 (\varphi^{\frac{1}{p}} u)(t) |f'(t)| dt.$$

By this inequality, in the usual way we deduce that

$$\|(R_m f)u\|_p \leq \frac{C}{m^{\frac{1}{p}}} E_{m-2}(f')_{u\varphi^{\frac{1}{p}}, 1} \quad (2.20)$$

where $E_{m-2}(f')_{u\varphi^{\frac{1}{p}}, 1} = \inf_{P \in \mathbb{P}_{m-2}} \|[f' - P]u\varphi^{\frac{1}{p}}\|_1$. Hence $\lim_m E_{m-2}(f')_{u\varphi^{\frac{1}{p}}, 1} = 0$ and the corollary is true for $r = 0$. Since (2.20) can be easily iterated, the proof is complete. \square

Finally we remark that the case $p = 1$ is still an open problem.

2.2. The L^∞ case

Let f be a continuous function on $[-1, 1]$. The following notation will be useful.

For $x, y \in [-1, 1]$ we set $[x, y; f] = \frac{f(y)-f(x)}{y-x}$ and for $x, y, z \in [-1, 1]$ we define the second divided difference of f by $[x, y, z; f] = \frac{[y, z; f] - [x, y; f]}{z-x}$.

If for any $x, y, z \in [-1, 1]$ it is true that $[x, y, z; f] \geq 0$, i.e. $[x, y; f]$ is nondecreasing with respect to the variable x (or y), then (by abuse of notation) we call f convex of order 1 (or simply convex) on $[-1, 1]$.

Now let

$$\mathcal{Z} = \begin{pmatrix} z_{11} & & & & \\ z_{21} & z_{22} & & & \\ z_{31} & z_{32} & z_{33} & & \\ z_{41} & z_{42} & z_{43} & z_{44} & \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

be an infinite matrix of knots on $[-1, 1]$ and let $q_m(x) = a \prod_{k=1}^m (x - z_{mk})$, $0 \neq a \in \mathbb{R}$, satisfy the condition

$$\sup_k \left| \sum_{i=1}^k \frac{1}{q'_m(z_{mi})} \right| \leq \frac{C}{m}, \quad C \neq C(m). \quad (2.21)$$

Moreover, for every fixed m , let $L_m(f, x)$ denote the Lagrange interpolation process based on the knots $z_1, z_2, \dots, z_m, z_k \equiv z_{mk}$. We can state the following:

Proposition 2.2. Let (2.21) be satisfied. Then for any convex function f on $[-1, 1]$ we have

$$|f(x) - L_m(f, x)| \leq \frac{\mathcal{C}}{m} |q_m(x)| [z_1, x, z_m; f] \quad (2.22)$$

where \mathcal{C} is a positive constant independent of m , x and f .

Proof. Since the Lagrange polynomial $L_m(f)$ can be written as

$$L_m(f, x) = \sum_{k=1}^m l_{m,k}(x) f(z_k) = \sum_{k=1}^m \frac{q_m(x)}{q'_m(z_k)} \frac{f(z_k)}{x - z_k},$$

we get

$$f(x) - L_m(f, x) = \sum_{k=1}^m \frac{q_m(x)}{q'_m(z_k)} \frac{f(x) - f(z_k)}{x - z_k} = q_m(x) \sum_{k=1}^m \frac{[z_k, x; f]}{q'_m(z_k)}.$$

By using the Abel transform we have

$$\begin{aligned} f(x) - L_m(f, x) &= -q_m(x) \sum_{k=1}^{m-1} \left(\sum_{i=1}^k \frac{1}{q'_m(z_i)} \right) ([z_{k+1}, x; f] - [z_k, x; f]) \\ &\quad + q_m(x) \sum_{k=1}^m \frac{[z_m, x; f]}{q'_m(z_k)}. \end{aligned}$$

Since $\sum_{k=1}^m \frac{q_m(x)}{q'_m(z_k)} = 0$ and f is convex, i.e. $[z_{k+1}, x; f] \geq [z_k, x; f]$, by (2.21), we get

$$\begin{aligned} |f(x) - L_m(f, x)| &\leq \frac{\mathcal{C}}{m} |q_m(x)| \sum_{k=1}^{m-1} ([z_{k+1}, x; f] - [z_k, x; f]) \\ &\leq \frac{2\mathcal{C}}{m} |q_m(x)| [z_1, x, z_m; f] \end{aligned}$$

and the proposition follows. \square

The estimate (2.22) explicitly shows the interpolation knots at the right-hand side, like in the well known formula

$$|f(x) - L_m(f, x)| \leq \frac{[z_1, z_2, \dots, z_m, x; f]}{m!} |q_m(x)|, \quad \forall x \in [-1, 1],$$

where $[z_1, z_2, \dots, z_m, x; f]$ stands for the m th divided difference of f . Moreover it allows us to prove the following theorem.

Theorem 2.2. Let f be continuous and $f^{(r)}$, $r \geq 1$ (eventually discontinuous), be of bounded variation. Then the following estimate:

$$|f(x) - L_m(f, x)| \leq \frac{\mathcal{C}}{m^r} |q_m(x)| \int_{-1}^1 \left(\sqrt{1-t^2} \right)^{r-1} |df^{(r)}(t)|, \quad (2.23)$$

holds true, with \mathcal{C} a positive constant independent of m , x and f .

We remark that, if $\sup_m \|q_m\|_\infty < \infty$, where $\|\cdot\|_\infty$ denotes the usual uniform norm, estimate (2.23) cannot be improved, since for any continuous function, with the r th derivative of bounded

variation, the limit condition (see formula (11), p. 436 in [14])

$$\lim_m m^r E_m(f) = \frac{\mu_r}{r!} \max_{x \in (-1,1)} |f^{(r)}(x)^+ - f^{(r)}(x)^-| \left(\sqrt{1-x^2} \right)^r,$$

holds true, where $E_m(f)$ denotes the error of best polynomial approximation in uniform norm and μ_r is a positive constant depending on r and defined in [14].

Proof. If f' is a bounded variation function, by (2.3) we get

$$|f(x) - L_m(f, x)| \leq \int_{-1}^1 |\Gamma_{t,1}(x) - L_m(\Gamma_{t,1}, x)| |df'(t)|. \quad (2.24)$$

Now, for any fixed t , $\Gamma_{t,1}(x)$ is convex and $0 \leq [z_1, x, z_m; \Gamma_{t,1}] \leq 1$. Therefore by (2.22) we obtain

$$|\Gamma_{t,1}(x) - L_m(\Gamma_{t,1}, x)| \leq \frac{C}{m} |q_m(x)|, \quad (2.25)$$

and hence (2.23) follows for $r = 1$.

Now let f'' be of bounded variation. Then, again by (2.3), we get

$$|f(x) - L_m(f, x)| \leq \frac{1}{2} \int_{-1}^1 |\Gamma_{t,2}(x) - L_m(\Gamma_{t,2}, x)| |df''(t)|. \quad (2.26)$$

Hence our next goal is to estimate $|\Gamma_{t,2}(x) - L_m(\Gamma_{t,2}, x)|$ for all $x, t \in [-1, 1]$. By (2.24) we have

$$|f(x) - L_m(f, x)| \leq \frac{C}{m} |q_m(x)| \int_{-1}^1 |f''(t)| dt,$$

and consequently

$$|f(x) - L_m(f, x)| \leq \frac{C}{m} |q_m(x)| E_{m-3}(f'')_1,$$

where $E_n(g)_1 = \inf_{P \in \mathbb{P}_n} \|g - P\|_1$ denotes the error of the best polynomial approximation in L^1 . Replacing, in the last estimate, f by $\Gamma_{t,2}$, we get

$$|\Gamma_{t,2}(x) - L_m(\Gamma_{t,2}, x)| \leq \frac{C}{m} |q_m(x)| E_{m-3}(\Gamma_{t,0})_1. \quad (2.27)$$

Since, for any fixed $t \in [-1, 1]$, we have

$$E_{m-3}(\Gamma_{t,0})_1 \leq \frac{C}{m} \sqrt{1-t^2}, \quad C \neq C(m, t), \quad (2.28)$$

using (2.27) with (2.28) and (2.26), we deduce

$$|f(x) - L_m(f, x)| \leq \frac{C}{m^2} |q_m(x)| \int_{-1}^1 \sqrt{1-t^2} |df''(t)|.$$

Therefore (2.23) holds true for $r = 2$.

The theorem follows by induction on r . \square

For the special case where $\mathcal{Z} = \mathcal{U} = \{(\cos \frac{k\pi}{m})_{k=0,\dots,m}\}_{m=1,2,\dots}$, (2.23) was essentially proved in [6, Corollary 2]. The proof was based on the representation of the second-kind Chebyshev

polynomials $U_m(x) = \frac{\sin((m+1)\arccos x)}{\sqrt{1-x^2}}$, in the complex plane and without the assumption (2.21), that on the other hand is trivially satisfied by system $\{U_m\}_m$.

In order to give some examples, that could be useful in some contexts, we consider the generalized Jacobi weight $w(x) = v^{\alpha,\beta}(x)|x|^\eta$, $\alpha, \beta > -1$, $\eta \geq 0$, and the corresponding sequence of orthonormal polynomials $\{p_m(w)\}_m$, with positive leading coefficients.

We can prove the following:

Lemma 2.1. *Let $x_1 < \dots < x_m$ be the zeros of $p_m(w)$. Then for any $k = 1, 2, \dots, m$, the estimate*

$$\left| \sum_{i=1}^k \frac{1}{p'_m(w, x_i)} \right| \leq \frac{C}{m}, \quad C \neq C(m, k), \quad (2.29)$$

holds true.

Proof. The proof is based on the equivalence [10]

$$|p'_m(w, x_i)| \sim \frac{m}{\varphi(x_i)} \frac{1}{(|x_i| + m^{-1})^{\frac{\eta}{2}} \sqrt{(v^{\alpha,\beta}\varphi)(x_i)}}, \quad i = 1, \dots, m, \quad (2.30)$$

where the constants in “ \sim ” are independent of m and i , and on the asymptotic formula proved in [1] (see also [17])

$$p'_m(w, x_i) = (-1)^i \sqrt{\frac{2}{\pi}} m \frac{1 + \frac{\mathcal{B}}{(|m/2-i|+1)}}{(1-x_i)^{\frac{\alpha}{2}+\frac{3}{4}}(1+x_i)^{\frac{\beta}{2}+\frac{3}{4}}|x_i|^{\frac{\eta}{2}}} \quad (2.31)$$

where $\mathcal{B} = \mathcal{B}(\alpha, \beta, \eta, i)$ is s.t. $|\mathcal{B}| \leq C \neq C(i)$. The formula (2.31) holds true for any x_i not “too close” to ± 1 and 0. In other words (2.31) can be used for the knots x_i defined as follows:

$$x_{k_0+1} < x_{k_0+2} < \dots < x_p < 0 < x_{2p} < \dots < x_{m-k_0-1}$$

where k_0 is fixed and small and $p = \lceil c \frac{m}{2} \rceil$, $0 < c < 1$, ($\lceil x \rceil$ denoting the integer part of x), is also fixed.

In order to evaluate $\left| \sum_{i=1}^k \frac{1}{p'_m(w, x_i)} \right|$, it is sufficient to assume that k is such that $x_k \leq 0$. For $x_k > 0$ we can proceed similarly, using the symmetry.

We will consider three sums:

$$\begin{aligned} \mathcal{S}_1 &:= \left| \sum_{i=1}^{k_0} \frac{1}{p'_m(w, x_i)} \right| \\ \mathcal{S}_2 &:= \left| \sum_{i=k_0+1}^p \frac{1}{p'_m(w, x_i)} \right| \\ \mathcal{S}_3 &:= \left| \sum_{x_p < x_i \leq 0} \frac{1}{p'_m(w, x_i)} \right|. \end{aligned}$$

In evaluating \mathcal{S}_1 and \mathcal{S}_3 we use (2.30) and, since $v^{\frac{\alpha}{2}+\frac{3}{4}, \frac{\beta}{2}+\frac{3}{4}}(x_i)|x_i|^{\frac{\eta}{2}} \leq C \neq C(i)$, we get

$$\mathcal{S}_1 \leq \frac{C}{m}, \quad \mathcal{S}_3 \leq \frac{C}{m}, \quad C \neq C(m).$$

Evaluate \mathcal{S}_2 . We use the following estimate (that is a consequence of (2.31); see [1, (35)–(39)]):

$$\left| \frac{p'_m(w, x_i) + p'_m(w, x_{i+1})}{p'_m(w, x_i)p'_m(w, x_{i+1})} \right| \sim \frac{(1-x_i)^{\frac{\alpha}{2}+\frac{1}{4}}(1+x_i)^{\frac{\beta}{2}+\frac{1}{4}}|x_i|^{\frac{\eta}{2}}}{m} \frac{\varphi(x_i)}{|m/2-i|+1}$$

with the constants in “ \sim ” independent of m and i . Since $k_0 < i \leq p$ and $p = \lfloor c \frac{m}{2} \rfloor$, $0 < c < 1$, we get $|m/2 - i| + 1 \geq Cm$, $C \neq C(m, i)$ and consequently $\frac{\varphi(x_i)}{|m/2-i|+1} \leq C\Delta x_i$, where $\Delta x_i := |x_{i+1} - x_i|$. Therefore we have

$$\begin{aligned} \mathcal{S}_2 &\leq \sum_{i=k_0+1}^{p-1} \left| \frac{1}{p'_m(w, x_i)} + \frac{1}{p'_m(w, x_{i+1})} \right| \\ &\leq \frac{C}{m} \sum_{i=k_0+1}^{p-1} \Delta x_i (1-x_i)^{\frac{\alpha}{2}+\frac{1}{4}} (1+x_i)^{\frac{\beta}{2}+\frac{1}{4}} |x_i|^{\frac{\eta}{2}} \\ &\leq \frac{C}{m} \int_{-1}^0 (1-x)^{\frac{\alpha}{2}+\frac{1}{4}} (1+x)^{\frac{\beta}{2}+\frac{1}{4}} |x|^{\frac{\eta}{2}} dx \leq \frac{C}{m}, \quad C \neq C(m). \end{aligned}$$

Thus the lemma easily follows by the previous estimates. Indeed if $k \leq k_0$ we use the estimate of \mathcal{S}_1 . Otherwise if $k \leq p$ then we can consider $\mathcal{S}_1 + \mathcal{S}_2$ and finally, for $k > p$, $x_k \leq 0$, we take $\mathcal{S}_1 + \mathcal{S}_2 + \mathcal{S}_3$. \square

We remark that (2.29) seems to be not true when the parameter η of the weight w is negative. Indeed, by (2.30) (since (2.31) cannot be used), and for every x_i “close” to 0, it follows that $\frac{1}{|p'_m(w, x_i)|} \geq \frac{C}{m^{1+\frac{\eta}{2}}}$, $\frac{\eta}{2} < 0$.

Obviously (2.29) holds even if w is a Jacobi weight. In this case the estimate

$$|f(x) - L_m(w, f, x)| \leq \frac{C}{m^r} |p_m(w, x)| \int_{-1}^1 \left(\sqrt{1-t^2} \right)^{r-1} |df^{(r)}(t)|,$$

where $C \neq C(m, f, x)$, is true. Anyway the convergence order will be $\mathcal{O}(m^{-r})$ only if $\sup_m \|p_m(w)\|_\infty < \infty$, i.e. when the Lebesgue constants $\|L_m(w)\| = \sup_{\|f\|_\infty=1} \|L_m(w, f)\|_\infty$ have the optimal order $\mathcal{O}(\log m)$.

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